MATC44 Week 11-12 Notes

1. Generating Functions:

- Developed by Euler in 1748.
- A very useful tool in combinatorics. The main idea is to turn combinatorial considerations into algebraic manipulations that can be done by a computer system.
- Consider this problem:
  Given any natural number n, we want to compute the number a_n of all possible ways to perform a given process that depends on n.

  The method of generating functions (GF) provides a new way to compute a_n by first computing the associated GF of a_n which is:

$$G_{a_n}(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \dots + a_n \cdot x^n + \dots$$
$$= \sum_{i=0}^{\infty} a_i \cdot x^i$$

- Note: $G_{a_n}(x) = G_{b_n}(x)$ iff $a_n = b_n$ for all $n \in \mathbb{N}$.

- Note: We will not consider issues of convergence. We will assume that x takes values s.t. the power series (the GF) are finite without worrying about the exact convergence interval.

- The main idea of the method is to compute a_n by first computing the associated GF $G_{a_n}(x)$. Once the GF has been found, then the seq is trivially computed since it is simply given by the coefficients of the monomials x^n for all n.

2. Algebraic Operations of GF:

a) Sum of GF:

- $G_{a_n}(x) + G_{b_n}(x) = G_{c_n}(x)$ where $c_n = a_n + b_n$

- Let a_n be associated with G_{a_n}.
  Let b_n be associated with G_{b_n}.
  Let $c_n = a_n + b_n$. Furthermore, let c_n be associated with G_{c_n}. Then:

$$
\begin{aligned}
G_{c_n}(x) &= \sum_{i=0}^{\infty} c_i \cdot x^i \\
&= \sum_{i=0}^{\infty} (a_i + b_i) \cdot x^i \\
&= \sum_{i=0}^{\infty} a_i \cdot x^i + b_i \cdot x^i \\
&= \sum_{i=0}^{\infty} a_i \cdot x^i + \sum_{i=0}^{\infty} b_i \cdot x^i \\
&= G_{a_n}(x) + G_{b_n}(x)
\end{aligned}
$$

b) Product of GF:

- $G_{a_n}(x) \cdot G_{b_n}(x) = G_{c_n}(x)$ where $c_n = a_0 \cdot b_n + a_1 \cdot b_{n-1} + a_2 \cdot b_{n-2} + \ldots + a_n \cdot b_0$

- Let a_n be associated with G_{a_n}.
  Let b_n be associated with G_{b_n}.

$$F_{a_n}(x) \cdot F_{b_n}(x) = (a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots) \cdot (b_0 + b_1 \cdot x + b_2 \cdot x^2 + \ldots)$$

$$= a_0 \cdot b_0 + a_0 \cdot b_1 \cdot x + a_0 \cdot b_2 \cdot x^2 + \ldots + a_1 \cdot x \cdot b_0 + a_1 \cdot b_1 \cdot x^2 + a_1 \cdot b_2 \cdot x^3 + \ldots + a_2 \cdot x^2 \cdot b_0 + a_2 \cdot b_1 \cdot x^3 + \ldots$$

$$= (a_0 \cdot b_0) + (a_0 \cdot b_1 + a_1 \cdot b_0) \cdot x + (a_0 \cdot b_2 + a_1 \cdot b_1 + a_2 \cdot b_0) x^2 + \ldots$$

$$= \sum_{k=0}^{\infty} (a_0 \cdot b_n + a_1 \cdot b_{n-1} + \ldots + a_n \cdot b_0) \cdot x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \cdot b_{n-k} \right) \cdot x^n$$

Hence, $c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$

c_n is known as the Cauchy Product.

3. Special Examples of GF

a) $a_n = 1$ for all n.

$$G_{a_n}(x) = 1 + x + x^2 + x^3 + \ldots$$
$$= \frac{1}{1-x}$$

Note: This formula follows from the identity $1 + x + x^2 + \ldots + x^n = \dfrac{1 - x^{n+1}}{1-x}$

and using the fact that $x^n \to 0$ as $x \to \infty$.

Furthermore, if $a_n = b$, then $G_{a_n}(x) = \dfrac{b}{1-x}$

and if $a_n = b^n$, then $G_{a_n}(x) = \dfrac{1}{1-bx}$

E.g. If $a_n = 2$, then $G_{a_n}(x) = \dfrac{2}{1-x}$

since $G_{a_n}(x) = 2 + 2x + 2x^2 + \ldots$
$$= 2(1 + x + x^2 + \ldots)$$

E.g. If $a_n = 2^n$, then $G_{a_n}(x) = 1 + 2x + 2^2 \cdot x^2 + 2^3 \cdot x^3 + \ldots + 2^n \cdot x^n + \ldots$

$$= \underbrace{1 + (2x) + (2x)^2 + \ldots + (2x)^n}$$

Note: These are the powers of $2x$.

$$= \frac{1}{1-2x}$$

b) If $a_k = \binom{n}{k}$ for $k \le n$ and $a_k = 0$ for $k > n$, then $G_{a_k}(x) = (1+x)^n$

I.e. $a_k = (\underbrace{\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}}_{\text{All} \neq 0}, \underbrace{\binom{n}{n+1}, \ldots}_{\text{All zero}})$

$$G_{a_k}(x) = \underbrace{\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n}_{\text{Finite power series}}$$

$$= (1+x)^n$$

c) If $a_k = 1$ for $k = n$ and $a_k = 0$ for all $k \neq n$, for some fixed n, then $G_{a_k}(x) = x^n$

4. Differentiation of GF:

- Recall that $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$.

  We know that if $a_n = 1$ for all n then $G_{a_n}(x) = 1 + x + x^2 + \ldots = \dfrac{1}{1-x}$.

  Differentiating both sides, we get:

  $$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + (n+1)x^n$$

  Hence, $\dfrac{1}{(1-x)^2}$ is the GF for the seq

  $a_n = n+1$

- Now, consider the fact that

$$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{2}{(1-x)^3}$$

By differentiating both sides of

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+\dots+(n+1)x^n,$$ we get

$$\frac{1}{(1-x)^3} = \frac{1}{2}\left(\quad 2+6x+\dots \right)$$

$$= 1+3x+\dots+\frac{(n+1)(n+2)}{2}x^n$$

Hence, $\frac{1}{(1-x)^3}$ is the GF for the seq

$$a_n = \frac{(n+1)(n+2)}{2}.$$

- Now, consider $C_n = n$.
  I.e. $C_1 = 1$, $C_2 = 2$, $C_3 = 3, \dots$
  Recall the following:
  a) If $a_n = 1$ for all n, then $G_{a_n}(x) = \frac{1}{1-x}$
  $\quad b_n \qquad\qquad\qquad G_{b_n}(x)$
  b) If $\quad = n+1$, then $\quad = \frac{1}{(1-x)^2}$

Hence, $C_n = b_n - a_n$ and $G_{C_n}(x) = G_{b_n}(x) - G_{a_n}(x)$.

$$G_{C_n}(x) = \frac{1}{(1-x)^2} - \frac{1}{1-x}$$

5. Converting Combinatorics to Algebra

- E.g. 1. Let a, b, c be 3 vars. a can be selected 1 or 2 times. b can be selected 0 or 1 times. c can be selected 0 or 1 times. Compute how many ways we can select 3 vars (no order). We also allow repetition.

Soln:

Let's consider a first.

$\underbrace{\text{a is selected}}_{\text{Combinatorial}}$ once corresponds to $\underbrace{a^1}_{\text{Algebraic}}$.

a is selected twice corresponds to a^2.

a selected once or twice corresponds to $a + a^2$.

Note: Or corresponds to $+$, And corresponds to $\cdot$ (multiplication).

Let's consider b now.

b is selected 0 times corresponds to b^0 or 1.

b is selected 1 time corresponds to b^1.

b is selected 0 or 1 time corresponds to $1 + b$.

c is the same as b.

c is selected 0 times corresponds to c^0 or 1.

c is selected 1 time corresponds to c^1.

c is selected 0 or 1 time corresponds to $1+c$.

To recap, here are the 3 expressions:

a) $a+a^2$

b) $1+b$

c) $1+c$

Now, we want to find out what is all the possibilities for a, b, c combined algebraically.

Hence, we need to consider all possibilities for a and all possibilities for b and all possibilities for c. As stated before, and corresponds to multiplication.

$$(a+a^2)\cdot(1+b)\cdot(1+c)$$
$$=(a+a^2+ab+a^2b)\cdot(1+c)$$
$$=\cancel{a}+\cancel{a^2}+\cancel{ab}+\underbrace{a^2b+\cancel{ac}+a^2c+abc}_{\text{These are the only valid possibilities.}}+\cancel{a^2bc}$$

Hence, there are 3 ways.

Note: This not only gave us the soln but also what the valid possibilities are.

To make it so that we only get the number of ways and not what the valid possibilities are, we do the following:

We will introduce a new var, x. (This is the major key.)

Furthermore, we will replace the 3 expressions:

a) $a^2 + a \rightarrow x + x^2$
b) $1 + b \rightarrow 1 + x$
c) $1 + c \rightarrow 1 + x$

We now get: $(x+x^2) \cdot (1+x) \cdot (1+x)$
$= (x + x^2 + x^2 + x^3)(1+x)$
$= x + x^2 + x^2 + x^3 + x^2 + x^3 + x^3 + x^4$
$= x + 3x^2 + 3x^3 + x^4$

x^i means selecting x i times.
Each of the terms above corresponds to 1 selection of k variables where k is the exponent.
I.e. There is 1 way to select 1 var. $(1 \cdot x)$

The coefficients tell us how many ways we can select k vars.

Since we want to select 3 vars, from above, we know that there are 3 ways to do so. Again, the answer is 3.

Note: Now, we don't know the valid possibilities.

- Summary: Suppose we have vars and restrictions for each of the vars. Then, we need to follow these steps:

Step 1. Introduce a new, independent var, say x. This var indicates that one of our original vars is selected, but we don't know which.

Step 2. x^k means that k of the vars are selected.
Note: x^0 means you are not selecting any vars.

Step 3. Obtain the polynomial (powerseries) that represents all the allowed possibilities for each of the vars.

Step 4. Multiply all polynomials to obtain all possibilities.

Step 5. Compute the coefficient of x^k. This will get you the answer.

Note: Order doesn't matter for all questions/examples of this type.

- E.g. 2. Consider the set $S = \{a, b, c\}$ s.t. the following restrictions applies:

a) a can be selected an even number of times. Note: 0 is an even number.

b) b can be selected an odd number of times.

c) c can be selected at most 10 times.

How many times can we select n of these vars. (No order). Repetition is allowed.

Soln:

The answer will be a seq, a_n.

Step 1.

We will introduce a var called x.

Step 2. and 3.

$a \rightarrow x^0 + x^2 + \ldots + x^{2k} + \ldots$

$b \rightarrow x + x^3 + x^5 + x^7 + \ldots + x^{2k+1} + \ldots$

$c \rightarrow x^0 + x^1 + x^2 + \ldots + x^{10}$

Step 4.

All poss $\rightarrow (1 + x^2 + x^4 + \ldots)(x + x^3 + \ldots)(1 + x + \ldots + x^{10})$

Let the above be denoted as $F(x)$.

Step 5.

Answer: a_n is the coefficient of x^n in $F(x)$.

If a^n is the coefficient of x^n in $F(x)$, then we get:

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$$

When we are solving these types of problems, we are first computing the G.F. and then extracting the coefficient out of the G.F. The coefficient gives us the answer.

This is the whole point of G.F. We first find the G.F. Then, we compute the seq a_n.

Main idea:

Desired: To compute a_n.

To do this:

1. Compute the G.F. $F_{a_n}(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$

2. Compute a_n.

- E.g. 3. Consider $S = \{a, b, c, d\}$ s.t.:

a) a can be selected at most 2 times.
b) b can be selected at most 1 time.
c) c can be selected at most 2 times.
d) d can be selected at most 1 time.

How many ways can we select n of these vars?

Soln:

$$\left.\begin{array}{l} a: 1+x+x^2 \\ b: 1+x \\ c: 1+x+x^2 \\ d: 1+x \end{array}\right\} \text{Associated PowerSeries}$$

$(1+x+x^2)^2 \cdot (1+x)^2$ } All possibilities

Answer: a_n, which is the coefficient of x^n in $(1+x)^2 \cdot (1+x+x^2)^2$.
Note: x^6 is the biggest term, above.
Hence, $a_n = 0$ if $n \geq 7$.

6. G.F. of $(0, a_0, \dots a_n)$:

- Consider $x \cdot F_{a_n}(x)$ where $F_{a_n}(x)$ is equal to $\sum_{n\geq 0} a_n \cdot x^n$. I.e. $F_{a_n}(x) = a_0 + a_1 x + \dots + a_n x^n$

Then, $x \cdot F_{a_n}(x) = \sum_{n \geq 0} a_{n-1} \cdot x^n$, where $a_{-1} = 0$.

I.e. $x \cdot F_{a_n}(x) = a_0 x + a_1 x^2 + \dots + a_n x^{n+1}$
This produces the list $(0, a_0, a_1, \dots, a_{n-1}, \dots)$.

Let $a_n = (a_0, a_1, \ldots a_n)$. $(F_{a_n}(x))$
Let $b_n = (0, a_0, a_1, \ldots a_n)$. $(x \cdot F_{a_n}(x))$
b_n is a right shift of a_n. When you right shift, you add 0 to the front.
Hence, $b_n = a_{n-1}$.

The above shows that multiplying the function with the var corresponds to a right shift.

- Consider $x^2 \cdot F_{a_n}(x)$. It is equal to $\sum_{n \geq 0} a_n \cdot x^{n+2}$ and produces this list of coefficients: $(0, 0, a_0, \ldots, a_{n-2}, \ldots)$.

- When you multiply $F_{a_n}(x)$ with x^n, then you right shift n times.

7. Multiplying $F_{a_n}(x)$ with $\frac{1}{x}$:
- If we divide $F_{a_n}(x)$ by x, then we get $\frac{a_0}{x} + a_1 + a_2 \cdot x + a_3 \cdot x^2 + \ldots + a_n \cdot x^{n-1}$.

- In general, this is NOT a GF. This is because we have the term $\frac{a_0}{x}$. Hence, only if $a_0 = 0$ does the above seq become a GF.

- Assuming $a_0 = 0$, then we get the list $(0, a_1, a_2, \ldots, a_n)$. $\frac{F_{a_n}(x)}{x} = a_1 + a_2 x + \ldots +$
$a_{n+1} x^n$.

If we let $F_{d_n}(x) = \frac{F_{a_n}(x)}{x}$, then

$d_n = (a_1, a_2, \dots, a_{n+1}, \dots)$.

n^{th} term

$d_n = a_{n+1},\ n \geq 0$

- When you divide your function by the var, you are shifting everything to the left.

- Summary of 6 and 7:

n^{th} term

$(a_0, a_1, a_2, \dots, a_n, \dots)$

Right Shift

Left Shift

n^{th} term

$(0, a_0, a_1, \dots, a_{n-1}, \dots)$

$GF = x \cdot F(x)$

$(a_1, a_2, \dots a_{n+1}, \dots)$

n^{th} term

$GF = \frac{F(x)}{x}$, if $a_0 = 0$

- Note: If you right shift n times and then left shift n times, you end up with the original function. Same applies if you left shift n times and then right shift n times.

8 Recurrence Relations:

- Will use the properties from 6 and 7 to solve them.

- E.g. 1. Solve: $a_{n+1} = 2a_n + f_n$, for some given seq$_n$ f_n, and a_0 is given. I.e. Find a_n relative to n.

Soln:
We will first find $F_{a_n}(x)$ using the info in the question. Then, we will compute the seq a_n.

Step 1. Compute $F_{a_n}(x)$ given the info in the question.

$a_{n+1} = 2a_n + f_n$
We know that the GF of a_{n+1} is equal to the GF of $2a_n + f_n$.

Consider a_n and a_{n+1}:
$a_n = (a_0, a_1, \ldots a_n, \ldots)$
$a_{n+1} = (a_1, a_2, \ldots a_{n+1}, \ldots)$
Hence, a_{n+1} is the left shift of a_n.

$$
\begin{aligned}
F_{a_{n+1}}(x) &= F_{(2a_n + f_n)}(x) \\
&= F_{(2a_n)}(x) + F_{f_n}(x) \\
&= 2 \cdot F_{a_n}(x) + F_{f_n}(x)
\end{aligned}
$$

Note: $F_{(2a_n)}(x)$
$= 2a_0 + 2a_1 + \ldots + 2a_n$
$= 2(a_0 + a_1 + \ldots + a_n)$
$= 2F_{a_n}(x)$

Note: Recall the left shift.

$F_{a_n}(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$

Let $GF_{a_n}(x)$ be the left shift of $F_{a_n}(x)$. Hence, $GF_{a_n}(x) = a_1 + a_2x + \ldots + a_nx^{n-1} + \ldots$

$$= \frac{(a_1x + a_2x^2 + \ldots + a_nx^n + \ldots)}{x}$$

$$= \frac{(-a_0 + \overbrace{a_0 + a_1x + \ldots + a_nx^n + \ldots}^{F_{a_n}(x)})}{x}$$

$$= \frac{-a_0}{x} + \frac{F_{a_n}(x)}{x}$$

Since we know that $F_{a_{n+1}}(x)$ is the left shift of $F_{a_n}(x)$, $F_{a_{n+1}}(x)$ is equal to $\frac{-a_0}{x} + \frac{F_{a_n}(x)}{x}$.

Putting it all together, we get:

$$\frac{F_{a_n}(x)}{x} - \frac{a_0}{x} = 2F_{a_n}(x) + F_{f_n}(x)$$

$$\rightarrow \left(\frac{1}{x} - 2\right)F_{a_n}(x) = F_{f_n}(x) + \frac{a_0}{x}$$

$$\rightarrow (1-2x)F_{a_n}(x) = a_0 + x \cdot F_{f_n}(x)$$

$$\rightarrow F_{a_n}(x) = \frac{a_0 + x \cdot F_{f_n}(x)}{1-2x}$$

We can find a_n from $F_{a_n}(x)$.

9. Partitions of Natural Numbers

- This is an application of GF.

- Def: A partition of the number n ($n \in \mathbb{N}$) is a collection of natural numbers s.t. their sum is equal to n.

- E.g. 1. Assume $n=3$. A partition of n is $1+2$.

Note: $1+2$ is the same partition as $2+1$. The order of the items of the partition doesn't matter.

Note: 0 is never part of a partition. The natural numbers that make up the partition must be greater than or equal to 1.

Note: Generally, the size of the partition is not constrained.

E.g. $1+2$ is a partition of 3, but 3 is also a partition of 3, and $1+1+1$ is also another partition of 3.

Note: Any partition of n must contain:

a) At least one number

b) At most n numbers

- E.g. 2. Assume $n=1$. The only partition is 1. There is 1 partition.

- E.g. 3. Assume $n=2$. The partitions are $1+1$ and 2. There are 2 partitions.

- E.g. 4. Assume $n=3$. The partitions are 3, 1+2, and 1+1+1. There are 3 partitions.

- E.g. 5. Assume $n=4$. The partitions are 4, 1+2+1, 2+2, 1+3, 1+1+1+1. There are 5 partitions.

- E.g. 6. Assume $n=5$. The partitions are

1. 1+1+1+1+1
2. 1+1+1+2
3. 1+2+2
4. 1+1+3
5. 1+4
6. 2+3
7. 5

There are 7 partitions.

- E.g. 7. Assume $n=6$. The partitions are

1. 1+1+1+1+1+1
2. 1+1+1+1+2
3. 1+1+1+3
4. 1+1+4
5. 1+5
6. 1+2+2+1
7. 1+2+3
8. 2+4
9. 2+2+2
10. 3+3
11. 6

These are the partitions of 6 that have a 1. Note that these are the partitions of 5 but with an additional 1.

Note: If you want to find the partitions of n start with 1 and go up to n.

Note: To find all partitions of n that includes the number x, find all partitions of n-x and add x to it. Be careful that you don't double count.

E.g. Consider n=6.

To find all partitions of 6 that includes a 1, find all partitions of 5 and add 1 to it.

1+1+1+1+1+1
1+1+1+2+1
1+2+2+1
1+1+3+1
1+4+1
2+3+1
5+1

These are all the partitions of 6 with a 1. The nums in orange represent the partitions of 5.

To find all the partitions of 6 that includes 2, find all partitions of 4 and add 2 to it.

1+1+1+1+2
1+2+1+2
2+2+2
1+3+2
4+2

These are the partitions of 6 that includes a 2. Notice that some of them have already been stated. Be careful not to double count.

- We will now find the G.F. of $P(n)$, where $P(n)$ is the number of partitions of n. Any partition of n consists of:

a) x_1 1's
b) x_2 2's
c) x_3 3's
$\vdots$
d) x_i i's

Given $x_1, x_2, \dots, x_i, \dots,$ we can form one unique partition.

E.g. Consider the following:

$$\left.\begin{array}{l} x_1 = 5 \\ x_2 = 2 \\ x_3 = 1 \\ x_k = 0,\ k \geq 4 \end{array}\right\} \overbrace{1+1+1+1+1}^{x_1=5} + \overbrace{2+2}^{x_2=2} + \overbrace{3}^{x_3=1}$$

This is a partition of 12.

Note: We can convert

x_1 1's
x_2 2's   into $\underbrace{1\cdot x_1 + 2\cdot x_2 + \dots + i\cdot x_i + \dots}$
$\vdots$
x_i i's   This becomes a system of linear eqns.

Note: $1\cdot x_1 + 2\cdot x_2 + \dots + i\cdot x_i + \dots = n$

- Any partition of n corresponds to a soln $(x_1, x_2, \ldots, x_n, \ldots)$ of the following linear eqn: $1 \cdot x_1 + 2 \cdot x_2 + \ldots + n \cdot x_n + \ldots = n$
If a_n = # of solns, then $a_n = P(n)$.
Note: $x_i \geq 0$, $x_i \in \mathbb{N}$
I.e. If a_n is the num of solns to $1 \cdot x_1 + 2 \cdot x_2 + \ldots + n \cdot x_n + (n+1) \cdot x_{n+1} + \ldots = n$, then $a_n = P(n)$.

- Consider the following:
$1 \cdot x_1 = 0, 1, 2, \ldots$
$2 \cdot x_2 = 0, 2, 4, \ldots$ (Multiples of 2)
$3 \cdot x_3 = 0, 3, 6, \ldots$ (Multiples of 3)
$i \cdot x_i = 0, i, 2i, \ldots$ (Multiples of i)
These are the restrictions of what $k \cdot x_k$ can be.

- Now, we will introduce a new var, x. It's meaning is that we will add 1 to an element.
Note: x^k means that we are adding k to the element.

- Going back to $k \cdot x_k$, we get the following:
$$1 \cdot x_1 = 1 + x + x^2 + \ldots = \frac{1}{1-x}$$

$$2 \cdot x_2 = 1 + x^2 + x^4 + \ldots$$

$$3 \cdot x_3 = 1 + x^3 + x^6 + \ldots$$

$$i \cdot x_i = 1 + x^i + x^{2i} + \ldots$$

- To get all the possible actions of $1 \cdot x_1 + 2 \cdot x_2 + \ldots + i \cdot x_i$, we will multiply them.

$$(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)\ldots(1+x^i+x^{2i}+\ldots)$$

$$= \prod_{i=1}^{\infty}(1+x^i+x^{2i}+x^{3i}+\ldots)$$

Note: To be able to find the GF, we first need to denote $P(0)$.
Recall that $F_{a_n}(x) = \sum_{n \geq 0} a_n \cdot x^n$, $a_n \geq 0$.

Hence, we will define $P(0)$ as 1. We choose to define $P(0)$ to be 1 because

$$\prod_{i=1}^{\infty}(\underbrace{1}+x^i+x^{2i}+\ldots)$$

Same as $1 \cdot x^0$. Hence, we want $P(0)=1$.
Want this to be $P(0)$.

The GF is the following:

$$F_{P(n)}(x) = \prod_{i=1}^{\infty}(1+x^i+x^{2i}+\ldots)$$

$$\rightarrow F_{P(n)}(x) = \prod_{i=1}^{\infty}\left(\frac{1}{1-x^i}\right)$$

10. Restricted Partitions

- E.g. 1. Let R_n be the num of all partitions of n that consists of 1 or 2 or 3 only. What is $F_{r_n}(x)$?

I.e. If $n=5$, $1+4$ is not a valid partition as it contains a number, 4, that is not 1, 2, or 3. All partitions of 5 that only contain 1, 2, or 3 are:

a) $1+1+1+1+1$
b) $1+1+3$
c) $1+2+2$
d) $2+3$
e) $1+1+1+2$

Soln:
As before, we will define x_i to be the number of all i's in the partition of n. Then, $1\cdot x_1 + 2\cdot x_2 + 3\cdot x_3 + \dots + i\cdot x_i + \dots = n$.
Now, we consider the restriction.
$x_4 = 0,\ x_5 = 0,\ \dots,\ x_i = 0$ for all $i \geq 4$.
Hence, the above eqn changes to
$1\cdot x_1 + 2\cdot x_2 + 3\cdot x_3 = n,\ x_1, x_2, x_3 \geq 0$

$$\left.\begin{aligned} 1\cdot x_1 &= 1+x+x^2+\dots \\ 2\cdot x_2 &= 1+x^2+x^4+\dots \\ 3\cdot x_3 &= 1+x^3+x^6+\dots \end{aligned}\right\} \text{Individual Power Series}$$

Hence, the GF is: $F_{r_n}(x) = (1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)$
$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)$$

- Distinct Partition:
  - Def: A distinct partition of n is a partition of n for which all terms are distinct.

  - E.g. If $n=5$, $1+4$ is a distinct partition but $1+1+3$ is not. $1+1+3$ has the number 1 twice.

- E.g. 2. Let d_n = # of all distinct partitions of n. Find $F_{d_n}(x)$.

Soln:
Any partition corresponds to a solution $(x_1, x_2, \ldots, x_i, \ldots)$ of $1 \cdot x_1 + 2 \cdot x_2 + \ldots + i \cdot x_i + \ldots = n$.
However, because of the restriction, $x_i = 0$ or 1.

$1 \cdot x_1 \to 1+x$
$2 \cdot x_2 \to 1+x^2$ (x_2 can be 0 or 1).
$\vdots$
$i \cdot x_i \to 1+x^i$

Hence, the GF is: $F_{d_n}(x) = (1+x)(1+x^2)(1+x^3)\ldots$
$$= \prod_{i=1}^{\infty} (1+x^i)$$

- Odd Partition:
  - Def: A partition is called odd if all its terms are odd.
  - E.g. If $n=5$, then $1+1+3$ is an odd partition but $2+3$ isn't.

- E.g. 3. Let O_n = # of all odd partitions of n. Find $F_{O_n}(x)$.

Soln:
Any partition corresponds to a solution $(x_1, x_2, \ldots, x_i, \ldots)$ of $1 \cdot x_1 + 2 \cdot x_2 + \ldots + i \cdot x_i = n$.
Because of our restriction, $x_2, x_4, \ldots, x_{2k}$ all equals 0.

$$1 \cdot x_1 = 1 + x + x^2 + \ldots$$
$$3 \cdot x_3 = 1 + x^3 + x^6 + \ldots$$
$$5 \cdot x_5 = 1 + x^5 + x^{10} + \ldots$$
$$\vdots$$
$$(2i+1) x_{2i+1} = 1 + x^{2i+1} + (x^{2i+1})^2 + \ldots$$

Hence, the GF is: $F_{O_n}(x) = \prod_{i=1}^{\infty} (1 + x^{2i+1} + (x^{2i+1})^2 + \ldots)$

$$= \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i+1}}$$

- Euler's Thm: The number of all odd partitions is equal to the number of all distinct partitions for all n.

Proof:

- To prove Euler's thm, we must show that the GF of all odd partitions equals to the GF of all distinct partitions.

- Recall that $F_{dn}(x) = \prod_{i=1}^{\infty} (1+x^i)$

- Recall that $F_{on}(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i+1}}$

Note: While it is not true that $1+x^i = \frac{1}{1-x^{2i+1}}$, $i \geq 1$, it is true that $(1+x)(1+x^2)\ldots = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\ldots$.

Furthermore, $1+x = \frac{1-x^2}{1-x}$.

We can rewrite $F_{dn}(x) = (1+x)(1+x^2)(1+x^3)\ldots$ as $\left(\frac{\cancel{1-x^2}}{1-x}\right)\left(\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\right)\left(\frac{\cancel{1-x^6}}{1-x^3}\right)\left(\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\right)\left(\frac{\cancel{1-x^{10}}}{1-x^5}\right)\ldots$

Note: The $(1-x^{2k})$ terms appear in both the numerator and denominator, so they will cancel out, leaving just the $(1+^{2k+1})$ terms. Furtlermore, the numerator of all the new fractions is 1.

Hence, we will be left with

$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\cdots$, which equals to

$\prod_{i=1}^{\infty} \frac{1}{1-x^{2i+1}}$, which is the GF of $F_{on}(x)$.